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Norm formulae for the Bethe Ansatz on root systems of small rank

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Abstract
The norms of the Bethe Ansatz eigenfunctions for the Lieb–Liniger quantum system of n Bosonic particles on a ring with pairwise repulsive delta potential interactions are given by a beautiful determinantal formula, first conjectured by Gaudin in the early seventies and then proven by Korepin about a decade later. Recently, E Emsiz formulated a similar conjecture generalizing the Gaudin–Korepin norm formula in terms of the root systems of complex simple Lie algebras. Here we confirm the validity of the conjecture in question for small root systems up to rank 3 (thus including the important test case of the exceptional root system G_2).

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1. Introduction

In a landmark paper, Lieb and Liniger showed in the early sixties that the quantum eigenvalue problem for n Bosonic particles on the circle with pairwise repulsive delta-potential interactions is exactly solvable by means of the Bethe Ansatz method [LL]. This fundamental discovery has since spurred a large amount of research concerning the properties of this integrable particle model [M, G2, KBI, S2]. Seminal results worth emphasizing in this context are (i) the proof of the orthogonality and completeness of the Bethe Ansatz eigenfunctions found by Lieb and Liniger [Do, YY] and (ii) the beautiful compact determinantal formula for the norms of the eigenfunctions in question, first conjectured by Gaudin [G2] and then proven by Korepin [K].

It was, furthermore, observed by Gaudin that the particle model of Lieb and Liniger admits an elegant mathematical generalization in terms of the root systems of complex simple Lie algebras, the solution of which is also amenable to the Bethe Ansatz approach [G1]. In a nutshell, the idea of this generalization is that the permutation symmetry of the model gets traded for the invariance with respect to an arbitrary crystallographic reflection group (i.e. a Weyl group). From this perspective, the original model of Lieb and Liniger is associated with the classical root systems of type A. The other classical root systems (types B, C and D) correspond to systems of particles configured symmetrically around the origin (with or without an external delta-potential supported at the origin), whereas the exceptional root systems (types E, F and G) give rise to physically somewhat less natural particle systems involving multi-particle interactions. For a detailed study of the properties of these generalized particle models with delta-potential interactions associated with root systems, the reader is referred to [GS, S1, G, HO, Di, EOS, E]. Here we confine ourselves to singling out that recently E Emsiz proposed a Gaudin–Korepin type conjectural determinantal formula for the norms of the Bethe-Ansatz eigenfunctions associated with root systems [E]. For the root systems of type A, this conjecture reduces to the celebrated Gaudin–Korepin norm formula. To date, it is only in this classical case that Emsiz's norm formula has actually been verified. The purpose of the present work is to provide highly desired further evidence for the correctness of the conjectural norm formula in question in the form of a direct check of all cases corresponding to small root systems up to rank 3 (thus including the important test case of the exceptional root system G_2).

The paper is organized as follows. We start by reviewing the Bethe Ansatz for the generalized particle models with delta-potential interactions associated with root systems in section 2. In section 3 we formulate E Emsiz' conjectural norm formula for the corresponding Bethe Ansatz eigenfunctions, and we indicate a computational scheme that permits checking the correctness of this conjecture for small root systems. The rest of the paper consists of a list of the cases for which the norm formula under consideration has been checked by means of this scheme (cf section 4).

2. Bethe Ansatz on root systems

In this section and the next one we will freely borrow concepts and standard properties from the theory of root systems [B, H]. Readers not so accustomed to this specific jargon may wish to skip straightaway to the concrete examples of section 4.

Let $\boldsymbol{E}$ be a real n-dimensional Euclidean space with inner product $\langle\cdot,\cdot\rangle$ and let $\mathbf{R}\subset\boldsymbol{E}$ denote an irreducible reduced crystallographic root system spanning $\boldsymbol{E}$. For a choice of positive roots $\mathbf{R}_+\subset\mathbf{R}$, we denote the maximal root by $\boldsymbol{\alpha}_0$ and the basis of simple roots by $\boldsymbol{\alpha}_1,\ldots,\boldsymbol{\alpha}_n$. The corresponding Weyl alcove

$$\boldsymbol{A}:=\{\mathbf{x}\in\boldsymbol{E}\mid 0<\langle\boldsymbol{\alpha},\mathbf{x}\rangle<1,\forall\boldsymbol{\alpha}\in\mathbf{R}_+\}\tag{2.1}$$

is bounded by the walls

$$\boldsymbol{E}_0:=\{\mathbf{x}\in\boldsymbol{E}\mid\langle\boldsymbol{\alpha}_0,\mathbf{x}\rangle=1\},\tag{2.2a}$$

$$\boldsymbol{E}_j:=\{\mathbf{x}\in\boldsymbol{E}\mid\langle\boldsymbol{\alpha}_j,\mathbf{x}\rangle=0\},\qquad j=1,\ldots,n.\tag{2.2b}$$

We will consider the following spectral problem for the Laplacian in the alcove $\boldsymbol{A}$ with repulsive boundary conditions at the walls:

$$-\Delta_{\mathbf{x}}\psi=E\psi,\qquad\mathbf{x}\in\boldsymbol{A},\tag{2.3a}$$

with

$$\left(\left\langle\nabla_{\mathbf{x}}\psi,\boldsymbol{\alpha}_0^\vee\right\rangle+g_{\alpha_0}\psi\right)\big|_{\mathbf{x}\in\boldsymbol{E}_0}=0,\tag{2.3b}$$

$$\left(\left\langle\nabla_{\mathbf{x}}\psi,\boldsymbol{\alpha}_j^\vee\right\rangle-g_{\alpha_j}\psi\right)\big|_{\mathbf{x}\in\boldsymbol{E}_j}=0,\qquad j=1,\ldots,n,\tag{2.3c}$$

where $\boldsymbol{\alpha}^\vee := 2\boldsymbol{\alpha}/\|\boldsymbol{\alpha}\|^2$ (with $\|\boldsymbol{\alpha}\| := \sqrt{\langle\boldsymbol{\alpha},\boldsymbol{\alpha}\rangle}$), $\Delta_{\mathbf{x}}$ and $\nabla_{\mathbf{x}}$ refer to the Laplacian and gradient, respectively, and g_α, $\boldsymbol{\alpha}\in\mathbf{R}$ denotes a real coupling parameter such that $g_\alpha = g_\beta$ if $\|\boldsymbol{\alpha}\| = \|\boldsymbol{\beta}\|$. More specifically, this means that the parameter in question takes the form

$$g_\alpha = \begin{cases} g_1 & \text{for} \quad \boldsymbol{\alpha} \text{ short} \\ g_2 & \text{for} \quad \boldsymbol{\alpha} \text{ long} \end{cases} \tag{2.4}$$

(where, by convention, all roots qualify as short when $\mathbf{R}$ is simply-laced).

Let $r_j : \boldsymbol{E} \to \boldsymbol{E}, 0 \leqslant j \leqslant n$ denote the orthogonal reflection in the wall $\boldsymbol{E}_j$, i.e. $r_0(\mathbf{x}) = \mathbf{x} - \langle\mathbf{x},\boldsymbol{\alpha}_0^\vee\rangle\boldsymbol{\alpha}_0 + \boldsymbol{\alpha}_0^\vee$ and $r_j(\mathbf{x}) = \mathbf{x} - \langle\mathbf{x},\boldsymbol{\alpha}_j^\vee\rangle\boldsymbol{\alpha}_j$ for $j = 1,\ldots,n$. The reflections $r_1,\ldots,r_n$ generate the Weyl group W associated with $\mathbf{R}$ and the reflections $r_0,\ldots,r_n$ generate the affine Weyl group $\hat{W}$. It was shown in [EOS] with the aid of Hecke-algebraic techniques that the Bethe Ansatz wavefunction

$$\Psi(\mathbf{x},\boldsymbol{\xi}) = \sum_{w\in W} \mathcal{C}(\boldsymbol{\xi}_w)\,\mathrm{e}^{\mathrm{i}\langle\boldsymbol{\xi}_w,\mathbf{x}\rangle} \tag{2.5a}$$

(where $\boldsymbol{\xi}_w := w(\boldsymbol{\xi})$) solves the eigenvalue problem in equations (2.3*a*)–(2.3*c*) with eigenvalue $E = \|\boldsymbol{\xi}\|^2$ when

$$\mathcal{C}(\boldsymbol{\xi}) = \prod_{\boldsymbol{\alpha}\in\mathbf{R}^+} \frac{\langle\boldsymbol{\xi},\boldsymbol{\alpha}^\vee\rangle - \mathrm{i}g_\alpha}{\langle\boldsymbol{\xi},\boldsymbol{\alpha}^\vee\rangle}, \tag{2.5b}$$

and the spectral parameter $\boldsymbol{\xi}\in\boldsymbol{E}$ satisfies the Bethe equations

$$\mathrm{e}^{\mathrm{i}\langle\boldsymbol{\xi},\boldsymbol{\beta}^\vee\rangle} = \left(\frac{\mathrm{i}g_\beta + \langle\boldsymbol{\xi},\boldsymbol{\beta}^\vee\rangle}{\mathrm{i}g_\beta - \langle\boldsymbol{\xi},\boldsymbol{\beta}^\vee\rangle}\right)^2 \prod_{\substack{\boldsymbol{\alpha}\in\mathbf{R} \\ \langle\boldsymbol{\alpha},\boldsymbol{\beta}^\vee\rangle=1}} \frac{\mathrm{i}g_\alpha + \langle\boldsymbol{\xi},\boldsymbol{\alpha}^\vee\rangle}{\mathrm{i}g_\alpha - \langle\boldsymbol{\xi},\boldsymbol{\alpha}^\vee\rangle}, \qquad \forall\boldsymbol{\beta}\in W(\boldsymbol{\alpha}_0) \tag{2.5c}$$

(where $W(\boldsymbol{\alpha})$ denotes the orbit of $\boldsymbol{\alpha}$ with respect to the action of the Weyl group W).

For the reader's convenience, we conclude this section with a short independent check of this result. Clearly $\Psi(\mathbf{x},\boldsymbol{\xi})$ (2.5*a*) satisfies the eigenvalue equation in equation (2.3*a*) with $E = \|\boldsymbol{\xi}\|^2$, since $-\Delta_{\mathbf{x}}\exp(\mathrm{i}\langle\boldsymbol{\xi}_w,\mathbf{x}\rangle) = \|\boldsymbol{\xi}_w\|^2\exp(\mathrm{i}\langle\boldsymbol{\xi}_w,\mathbf{x}\rangle) = \|\boldsymbol{\xi}\|^2\exp(\mathrm{i}\langle\boldsymbol{\xi}_w,\mathbf{x}\rangle)$. Furthermore, a short computation shows that for coefficients given by equation (2.5*b*) the wavefunction $\Psi(\mathbf{x},\boldsymbol{\xi})$ (2.5*a*) satisfies the boundary conditions in equation (2.3*c*). Indeed, for $\mathbf{x}\in\boldsymbol{E}_j$ one has that

$$\begin{aligned}
\langle\nabla_{\mathbf{x}}\Psi,\boldsymbol{\alpha}_j^\vee\rangle &= \sum_{w\in W}\left(\prod_{\boldsymbol{\alpha}\in\mathbf{R}_+}\frac{\langle\boldsymbol{\xi}_w,\boldsymbol{\alpha}^\vee\rangle - \mathrm{i}g_\alpha}{\langle\boldsymbol{\xi}_w,\boldsymbol{\alpha}^\vee\rangle}\right)\mathrm{i}\langle\boldsymbol{\xi}_w,\boldsymbol{\alpha}_j^\vee\rangle\,\mathrm{e}^{\mathrm{i}\langle\boldsymbol{\xi}_w,\mathbf{x}\rangle} \\
&= \sum_{w\in W}\left(g_{\alpha_j} + \mathrm{i}\langle\boldsymbol{\xi}_w,\boldsymbol{\alpha}_j^\vee\rangle\right)\left(\prod_{\substack{\boldsymbol{\alpha}\in\mathbf{R}_+ \\ \boldsymbol{\alpha}\neq\boldsymbol{\alpha}_j}}\frac{\langle\boldsymbol{\xi}_w,\boldsymbol{\alpha}^\vee\rangle - \mathrm{i}g_\alpha}{\langle\boldsymbol{\xi}_w,\boldsymbol{\alpha}^\vee\rangle}\right)\mathrm{e}^{\mathrm{i}\langle\boldsymbol{\xi}_w,\mathbf{x}\rangle} \\
&\overset{\text{(i)}}{=} g_{\alpha_j}\sum_{w\in W}\left(\prod_{\substack{\boldsymbol{\alpha}\in\mathbf{R}_+ \\ \boldsymbol{\alpha}\neq\boldsymbol{\alpha}_j}}\frac{\langle\boldsymbol{\xi}_w,\boldsymbol{\alpha}^\vee\rangle - \mathrm{i}g_\alpha}{\langle\boldsymbol{\xi}_w,\boldsymbol{\alpha}^\vee\rangle}\right)\mathrm{e}^{\mathrm{i}\langle\boldsymbol{\xi}_w,\mathbf{x}\rangle} \\
&\overset{\text{(ii)}}{=} g_{\alpha_j}\sum_{w\in W}\left(1 - \frac{\mathrm{i}g_{\alpha_j}}{\langle\boldsymbol{\xi}_w,\boldsymbol{\alpha}_j^\vee\rangle}\right)\left(\prod_{\substack{\boldsymbol{\alpha}\in\mathbf{R}_+ \\ \boldsymbol{\alpha}\neq\boldsymbol{\alpha}_j}}\frac{\langle\boldsymbol{\xi}_w,\boldsymbol{\alpha}^\vee\rangle - \mathrm{i}g_\alpha}{\langle\boldsymbol{\xi}_w,\boldsymbol{\alpha}^\vee\rangle}\right)\mathrm{e}^{\mathrm{i}\langle\boldsymbol{\xi}_w,\mathbf{x}\rangle} \\
&= g_{\alpha_j}\sum_{w\in W}\left(\prod_{\boldsymbol{\alpha}\in\mathbf{R}_+}\frac{\langle\boldsymbol{\xi}_w,\boldsymbol{\alpha}^\vee\rangle - \mathrm{i}g_\alpha}{\langle\boldsymbol{\xi}_w,\boldsymbol{\alpha}^\vee\rangle}\right)\mathrm{e}^{\mathrm{i}\langle\boldsymbol{\xi}_w,\mathbf{x}\rangle} \\
&= g_{\alpha_j}\Psi,
\end{aligned}$$

where, in steps (i) and (ii), one exploits that $\prod_{\substack{\alpha\in\mathbf{R}_+\\ \alpha\neq\alpha_j}} \frac{\langle\boldsymbol{\xi}_w,\boldsymbol{\alpha}^\vee\rangle-\mathrm{i}g_\alpha}{\langle\boldsymbol{\xi}_w,\boldsymbol{\alpha}^\vee\rangle}$ and $\langle\boldsymbol{\xi}_w,\boldsymbol{\alpha}_j^\vee\rangle$ are symmetric and skew-symmetric, respectively, with respect to the action of r_j on $\boldsymbol{\xi}_w$, combined with the symmetry $\langle\mathbf{x}, r_j(\boldsymbol{\xi}_w)\rangle = \langle\mathbf{x},\boldsymbol{\xi}_w\rangle$ (since $r_j(\mathbf{x}) = \mathbf{x}$ for $\mathbf{x}\in\boldsymbol{E}_j$). Similarly, the boundary condition in equation (2.3*b*) requires that for $\mathbf{x}\in\boldsymbol{E}_0$

$$\sum_{w\in W}\left(g_{\alpha_0}+\mathrm{i}\langle\boldsymbol{\xi}_w,\boldsymbol{\alpha}_0^\vee\rangle\right)\mathcal{C}(\boldsymbol{\xi}_w)\,\mathrm{e}^{\mathrm{i}\langle\boldsymbol{\xi}_w,\mathbf{x}\rangle}=0,$$

or equivalently

$$\sum_{\substack{w\in W\\ w^{-1}(\alpha_0)\in\mathbf{R}_+}}\left(\left(g_{\alpha_0}+\mathrm{i}\langle\boldsymbol{\xi}_w,\boldsymbol{\alpha}_0^\vee\rangle\right)\mathcal{C}(\boldsymbol{\xi}_w)\,\mathrm{e}^{\mathrm{i}\langle\boldsymbol{\xi}_w,\mathbf{x}\rangle}+\left(g_{\alpha_0}+\mathrm{i}\langle r_0^{(0)}(\boldsymbol{\xi}_w),\boldsymbol{\alpha}_0^\vee\rangle\right)\mathcal{C}\left(r_0^{(0)}(\boldsymbol{\xi}_w)\right)\mathrm{e}^{\mathrm{i}\langle r_0^{(0)}(\boldsymbol{\xi}_w),\mathbf{x}\rangle}\right)=0,$$

where $r_0^{(0)}\in W$ denotes the orthogonal reflection $r_0^{(0)}(\mathbf{x})=\mathbf{x}-\langle\mathbf{x},\boldsymbol{\alpha}_0^\vee\rangle\boldsymbol{\alpha}_0$. The latter equation translates to

$$\sum_{\substack{w\in W\\ w^{-1}(\alpha_0)\in\mathbf{R}_+}}\left(\left(g_{\alpha_0}+\mathrm{i}\langle\boldsymbol{\xi}_w,\boldsymbol{\alpha}_0^\vee\rangle\right)\mathcal{C}(\boldsymbol{\xi}_w)+\left(g_{\alpha_0}-\mathrm{i}\langle\boldsymbol{\xi}_w,\boldsymbol{\alpha}_0^\vee\rangle\right)\mathcal{C}\left(r_0^{(0)}(\boldsymbol{\xi}_w)\right)\mathrm{e}^{-\mathrm{i}\langle\boldsymbol{\xi}_w,\boldsymbol{\alpha}_0^\vee\rangle}\right)\mathrm{e}^{\mathrm{i}\langle\boldsymbol{\xi}_w,\mathbf{x}\rangle}=0,$$

which is satisfied if

$$\left(g_{\alpha_0}+\mathrm{i}\langle\boldsymbol{\xi}_w,\boldsymbol{\alpha}_0^\vee\rangle\right)\mathcal{C}(\boldsymbol{\xi}_w)+\left(g_{\alpha_0}-\mathrm{i}\langle\boldsymbol{\xi}_w,\boldsymbol{\alpha}_0^\vee\rangle\right)\mathcal{C}\left(r_0^{(0)}(\boldsymbol{\xi}_w)\right)\mathrm{e}^{-\mathrm{i}\langle\boldsymbol{\xi}_w,\boldsymbol{\alpha}_0^\vee\rangle}=0$$

for all $w\in W$, or equivalently (assuming $\mathcal{C}(\boldsymbol{\xi}_w)\neq 0$)

$$\mathrm{e}^{\mathrm{i}\langle\boldsymbol{\xi},\boldsymbol{\beta}^\vee\rangle}=-\frac{\mathcal{C}\left(r_0^{(0)}(\boldsymbol{\xi}_w)\right)}{\mathcal{C}(\boldsymbol{\xi}_w)}\frac{g_{\alpha_0}-\mathrm{i}\langle\boldsymbol{\xi}_w,\boldsymbol{\alpha}_0^\vee\rangle}{g_{\alpha_0}+\mathrm{i}\langle\boldsymbol{\xi}_w,\boldsymbol{\alpha}_0^\vee\rangle},\qquad\forall w\in W,$$

where $\boldsymbol{\beta}:=w^{-1}(\boldsymbol{\alpha}_0)$. The Bethe equations in equation (2.5*c*) now follow upon inserting

$$\frac{\mathcal{C}\left(r_0^{(0)}(\boldsymbol{\xi}_w)\right)}{\mathcal{C}(\boldsymbol{\xi}_w)}=-\prod_{\substack{\alpha\in\mathbf{R}_+\\ \langle\alpha,\alpha_0^\vee\rangle>0}}\frac{\mathrm{i}g_\alpha+\langle\boldsymbol{\xi}_w,\boldsymbol{\alpha}^\vee\rangle}{\mathrm{i}g_\alpha-\langle\boldsymbol{\xi}_w,\boldsymbol{\alpha}^\vee\rangle}=-\prod_{\substack{\alpha\in\mathbf{R}\\ \langle\alpha,\beta^\vee\rangle>0}}\frac{\mathrm{i}g_\alpha+\langle\boldsymbol{\xi},\boldsymbol{\alpha}^\vee\rangle}{\mathrm{i}g_\alpha-\langle\boldsymbol{\xi},\boldsymbol{\alpha}^\vee\rangle}.$$

3. Norm formulae

Notes. From now on we will always assume that the coupling parameter is *repulsive*: $g_\alpha>0,\forall\boldsymbol{\alpha}\in\mathbf{R}$.

Let $\boldsymbol{\xi}_\lambda$ denote the unique global minimum of the strictly convex function

$$V_\lambda(\boldsymbol{\xi}):=\frac{1}{2}\langle\boldsymbol{\xi},\boldsymbol{\xi}\rangle+\sum_{\alpha\in\mathbf{R}_+}\|\boldsymbol{\alpha}\|^2\int_0^{\langle\boldsymbol{\xi},\boldsymbol{\alpha}^\vee\rangle}\arctan\left(\frac{x}{g_\alpha}\right)\mathrm{d}x-2\pi\langle\boldsymbol{\rho}+\boldsymbol{\lambda},\boldsymbol{\xi}\rangle,\tag{3.1}$$

with $\boldsymbol{\rho}:=\frac{1}{2}\sum_{\alpha\in\mathbf{R}_+}\boldsymbol{\alpha}$ and $\boldsymbol{\lambda}$ taken from the cone of dominant weights

$$\Lambda:=\{\boldsymbol{\lambda}\in\boldsymbol{E}\mid\langle\boldsymbol{\lambda},\boldsymbol{\alpha}^\vee\rangle\in\mathbb{Z}_{\geqslant 0},\forall\boldsymbol{\alpha}\in\mathbf{R}_+\}.\tag{3.2}$$

It was shown in [EOS] that $\boldsymbol{\xi}_\lambda$ solves the Bethe equations in equation (2.5*c*) (indeed, the critical equation $\nabla_\xi V_\lambda(\boldsymbol{\xi})=0$ amounts to the Bethe equation (2.5*c*) upon exponentiation) and that this minimum is assumed inside the Weyl chamber

$$\boldsymbol{C}:=\{\boldsymbol{\xi}\in\boldsymbol{E}\mid\langle\boldsymbol{\alpha},\boldsymbol{\xi}\rangle>0,\forall\boldsymbol{\alpha}\in\mathbf{R}_+\}.\tag{3.3}$$

The Bethe wavefunctions $\Psi(\boldsymbol{\xi}_\lambda,\mathbf{x})$, $\boldsymbol{\lambda}\in\Lambda$, turn out to be complete in the sense that they span a dense subspace of $L^2(\boldsymbol{A},\mathrm{d}\mathbf{x})$ [E]. In *loc. cit.* E Emsiz, moreover, conjectured the following formula for the normalization constants of the Bethe wavefunctions at issue.

Conjecture ([E]). *The quadratic norm of the Bethe wavefunction* $\Psi(\boldsymbol{\xi}_{\boldsymbol{\lambda}}, \mathbf{x})$, $\boldsymbol{\lambda} \in \Lambda$, *is given by*

$$\frac{1}{|W|\mathrm{Vol}(\boldsymbol{A})} \int_{\boldsymbol{A}} |\Psi(\boldsymbol{\xi}_{\boldsymbol{\lambda}}, \mathbf{x})|^2 \,\mathrm{d}\mathbf{x} = \mathcal{C}(\boldsymbol{\xi}_{\boldsymbol{\lambda}})\mathcal{C}(-\boldsymbol{\xi}_{\boldsymbol{\lambda}}) \det \mathcal{H}(\boldsymbol{\xi}_{\boldsymbol{\lambda}}), \tag{3.4a}$$

where $|W|$ *denotes the order of the Weyl group,* $\mathrm{Vol}(\boldsymbol{A}) := \int_{\boldsymbol{A}} 1\mathrm{d}\mathbf{x}$, $\mathcal{C}(\boldsymbol{\xi})$ *is given by equation (2.5b) and* $\mathcal{H}(\boldsymbol{\xi})$ *stands for the Hessian of* $V_{\boldsymbol{\lambda}}(\boldsymbol{\xi})$ *(3.1), i.e.* $\langle \mathcal{H}(\boldsymbol{\xi})\boldsymbol{\zeta}, \boldsymbol{\eta}\rangle = \mathcal{H}_{\zeta\eta}(\boldsymbol{\xi})$ *with*

$$\mathcal{H}_{\zeta\eta}(\boldsymbol{\xi}) := \frac{\partial^2 V_{\boldsymbol{\lambda}}(\boldsymbol{\xi})}{\partial \boldsymbol{\zeta} \partial \boldsymbol{\eta}} = \langle \boldsymbol{\zeta}, \boldsymbol{\eta} \rangle + \sum_{\boldsymbol{\alpha} \in \mathbf{R}_+} g_{\alpha} \|\boldsymbol{\alpha}\|^2 \frac{\langle \boldsymbol{\zeta}, \boldsymbol{\alpha}^\vee \rangle \langle \boldsymbol{\eta}, \boldsymbol{\alpha}^\vee \rangle}{g_\alpha^2 + \langle \boldsymbol{\xi}, \boldsymbol{\alpha}^\vee \rangle^2} \tag{3.4b}$$

(here $\frac{\partial V_{\boldsymbol{\lambda}}(\boldsymbol{\xi})}{\partial \zeta} := \langle \nabla_{\xi} V_{\boldsymbol{\lambda}}(\boldsymbol{\xi}), \zeta \rangle$*).*

A direct computation of the quadratic norm entails that

$$\frac{1}{|W|\mathrm{Vol}(\boldsymbol{A})} \int_{\boldsymbol{A}} |\Psi(\boldsymbol{\xi}, \mathbf{x})|^2 \,\mathrm{d}\mathbf{x} = \mathcal{C}(\boldsymbol{\xi})\mathcal{C}(-\boldsymbol{\xi}) \sum_{w,w' \in W} \frac{\mathcal{C}(\boldsymbol{\xi}_w)}{\mathcal{C}(\boldsymbol{\xi}_{w'})} \mathcal{R}(\boldsymbol{\xi}_w - \boldsymbol{\xi}_{w'}), \tag{3.5a}$$

where

$$\mathcal{R}(\boldsymbol{\xi}) := \frac{1}{|W|\mathrm{Vol}(\boldsymbol{A})} \int_{\boldsymbol{A}} \mathrm{e}^{\mathrm{i}\langle \boldsymbol{\xi}, \mathbf{x} \rangle} \,\mathrm{d}\mathbf{x}. \tag{3.5b}$$

To prove the normalization conjecture it thus suffices to check that

$$\sum_{w,w' \in W} \frac{\mathcal{C}(\boldsymbol{\xi}_w)}{\mathcal{C}(\boldsymbol{\xi}_{w'})} \mathcal{R}(\boldsymbol{\xi}_w - \boldsymbol{\xi}_{w'}) = \det \mathcal{H}(\boldsymbol{\xi}), \tag{3.6}$$

provided that the spectral parameter $\boldsymbol{\xi}$ satisfies the Bethe equations (2.5c).

Upon expressing $\boldsymbol{\xi}$ and $\boldsymbol{\alpha}_0$ in the basis of the simple roots: $\boldsymbol{\xi} = \hat{\xi}_1 \boldsymbol{\alpha}_1 + \cdots + \hat{\xi}_n \boldsymbol{\alpha}_n$ and $\boldsymbol{\alpha}_0 = \hat{m}_1 \boldsymbol{\alpha}_1 + \cdots + \hat{m}_n \boldsymbol{\alpha}_n$, and writing $\mathbf{x}$ in the dual basis (of the fundamental coweights) it is not very difficult to infer that

$$\mathcal{R}(\boldsymbol{\xi}) = \frac{1}{2\pi \mathrm{i}\, \mathrm{Ind}(\mathbf{R})} \oint_C \frac{(\mathrm{e}^z - 1)}{(\hat{m}_1 z - \mathrm{i}\hat{\xi}_1) \cdots (\hat{m}_n z - \mathrm{i}\hat{\xi}_n)} \frac{\mathrm{d}z}{z}, \tag{3.7}$$

where $C \subset \mathbb{C}$ denotes a positively oriented (Jordan) contour looping (once) around the poles $\mathrm{i}\hat{\xi}_j/\hat{m}_j$, $j = 1, \ldots, n$, and $\mathrm{Ind}(\mathbf{R})$ refers to the connection index of the root system $\mathbf{R}$ (i.e. the order of the weight lattice modulo the root lattice). Indeed, for generic $\boldsymbol{\xi}$ one has that

$$\begin{aligned}
\frac{1}{\mathrm{Vol}(\boldsymbol{A})} \int_{\boldsymbol{A}} \mathrm{e}^{\mathrm{i}\langle \boldsymbol{\xi}, \mathbf{x} \rangle} \,\mathrm{d}\mathbf{x} &= \frac{\int_{\substack{\hat{m}_1 x_1 + \cdots + \hat{m}_n x_n \leqslant 1 \\ x_1, \ldots, x_n \geqslant 0}} \exp(\mathrm{i}\hat{\xi}_1 x_1 + \cdots + \mathrm{i}\hat{\xi}_n x_x) \,\mathrm{d}x_1 \cdots \mathrm{d}x_n}{\int_{\substack{\hat{m}_1 x_1 + \cdots + \hat{m}_n x_n \leqslant 1 \\ x_1, \ldots, x_n \geqslant 0}} 1 \,\mathrm{d}x_1 \cdots \mathrm{d}x_n} \\
&= n! \sum_{1 \leqslant j \leqslant n} \frac{\hat{m}_j}{\mathrm{i}\hat{\xi}_j} \left(\exp\left(\frac{\mathrm{i}\hat{\xi}_j}{\hat{m}_j} \right) - 1 \right) \prod_{\substack{1 \leqslant k \leqslant n \\ k \neq j}} \left(\frac{\mathrm{i}\hat{\xi}_j}{\hat{m}_j} - \frac{\mathrm{i}\hat{\xi}_k}{\hat{m}_k} \right)^{-1} \\
&= \frac{\hat{m}_1 \cdots \hat{m}_n n!}{2\pi i} \oint_C \frac{(\mathrm{e}^z - 1)}{(\hat{m}_1 z - \mathrm{i}\hat{\xi}_1) \cdots (\hat{m}_n z - \mathrm{i}\hat{\xi}_n)} \frac{\mathrm{d}z}{z},
\end{aligned}$$

from which equation (3.7) follows (first for generic $\boldsymbol{\xi}$ and then for general $\boldsymbol{\xi}$ by continuity) upon invoking the well-known formula $|W| = n! \hat{m}_1 \cdots \hat{m}_n \mathrm{Ind}(\mathbf{R})$. (The evaluation of the multivariate integrals in the numerator and the denominator of the expression on the second line is readily performed via a rescaling of the variables $x_j \to x_j/\hat{m}_j$, $j = 1, \ldots, n$ followed by an induction argument involving the dimension n, thus leading to the expression on the third line.)

The idea for proving the norm formula of the conjecture is now (i) to plug equation (3.7) into the lhs of equation (3.6), (ii) to eliminate all exponentials from the resulting expression with the aid of the Bethe equations (2.5*c*) and (iii) to check that the final result simplifies to the rhs of equation (3.6) as an identity in $\boldsymbol{\xi}$.

It is instructive to detail the relevant computations explicitly for the trivial example of a root system of rank $n = 1$. In this case $\mathbf{E} = \mathbb{R}$ and $\mathbf{R} = \{a, -a\}$ (with $a \neq 0$). The Bethe wavefunction (2.5*a*) and the Bethe equation (2.5*c*) become, in this situation,

$$\Psi(\xi, x) = c(\xi)\mathrm{e}^{\mathrm{i}\xi x} + c(-\xi)\,\mathrm{e}^{-\mathrm{i}\xi x}, \qquad \mathrm{e}^{2\mathrm{i}\xi/a} = \left(\frac{c(-\xi)}{c(\xi)}\right)^2, \tag{3.8}$$

with $c(\xi) = (1 - \frac{\mathrm{i}ag}{2\xi})$. The norm formula in the conjecture now reads

$$\frac{a}{2}\int_0^{1/a} |\Psi(\xi_\ell, x)|^2\,\mathrm{d}x = c(\xi_\ell)c(-\xi_\ell)\left(1 + \frac{4a^2g}{a^2g^2 + 4\xi_\ell^2}\right), \tag{3.9a}$$

where ξ_ℓ denotes the global minimum of the function

$$V_\ell(\xi) = \frac{\xi^2}{2} + a^2\int_0^{2\xi/a} \arctan\left(\frac{x}{g}\right)\mathrm{d}x - \pi a\xi(\ell + 1), \tag{3.9b}$$

$\ell = 0, 1, 2, \ldots.$ (Note, in this connection, that the critical equation $V_\ell'(\xi) = 0$ for the minimum passes over into the Bethe equation in equation (3.8) upon exponentiation.) The direct evaluation of the integral on the other hand yields (cf equations (3.5*a*) and (3.5*b*))

$$\frac{a}{2}\int_0^{1/a} |\Psi(\xi, x)|^2\,\mathrm{d}x = c(\xi)c(-\xi)\left(1 + \frac{c(\xi)}{c(-\xi)}\mathcal{R}(2\xi) + \frac{c(-\xi)}{c(\xi)}\mathcal{R}(-2\xi)\right), \tag{3.10a}$$

with

$$\mathcal{R}(\xi) = \frac{a}{2}\int_0^{1/a} \mathrm{e}^{\mathrm{i}\xi x}\,\mathrm{d}x = \frac{1}{4\pi\mathrm{i}}\oint_C \left(\frac{\mathrm{e}^z - 1}{z - i\xi/a}\right)\frac{\mathrm{d}z}{z} = \frac{\mathrm{e}^{\mathrm{i}\xi/a} - 1}{2\mathrm{i}\xi/a}. \tag{3.10b}$$

Substitution of $\mathcal{R}(\xi)$ (3.10*b*) into the factor between brackets on the rhs of equation (3.10*a*) and elimination of the exponentials $\mathrm{e}^{\pm 2\mathrm{i}\xi/a}$ by means of the Bethe equation in equation (3.8) then entails

$$1 + \frac{\frac{c(-\xi)}{c(\xi)} - \frac{c(\xi)}{c(-\xi)}}{2\mathrm{i}\xi/a},$$

which indeed simplifies to $V_\ell''(\xi) = 1 + \frac{4a^2g}{a^2g^2+4\xi^2}$, thus confirming the norm formula of the conjecture in this trivial example.

4. Explicit integrals

From the above analysis it is clear that use of formula (3.7) and the Bethe equations (2.5*c*) reduces the proof of the normalization conjecture to verifying the algebraic identity in equation (3.6). For larger root systems the computations verifying this elementary identity are extremely tedious, and we have in fact only been able to perform the check in question for small root systems up to rank $n = 3$ with the aid of computer algebra.

Theorem. *The normalization conjecture in section 3 is valid for all (reduced irreducible) crystallographic root systems of rank* $n \leqslant 3$.

In this section, we present an explicit list of the corresponding normalization formulae. The list at issue is readily obtained upon converting formulae for general root systems from sections 2 and 3 into explicit formulae for concrete root systems via Bourbaki's root system tables [B]. Throughout we employ *positive* parameters g_1, g_2 in accordance with equation (2.4); when the root system is simply-laced the index is suppressed: $g_1 = g_2 = g$. Moreover, depending on what is most convenient, we will alternate the following standard notations for vectors in $\mathbb{R}^m$: $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_m) = \xi_1 \mathbf{e}_1 + \cdots + \xi_m \mathbf{e}_m$ (where $\mathbf{e}_1, \ldots, \mathbf{e}_m$ denotes the standard basis of $\mathbb{R}^m$).

4.1. The case A_n $(n \geqslant 1)$

As remarked in [E], this case reproduces the celebrated results of Lieb and Liniger [LL] (Bethe wavefunction and Bethe equations), of Yang and Yang [YY] (solution of the Bethe equations) and of Gaudin [G2] and Korepin [K] (norm formulae) in the center-of-mass frame. It is included here for completeness only.

The Bethe Ansatz wavefunction (2.5*a*) and the Bethe equations (2.5*c*) amount, respectively, to

$$\begin{aligned} \Psi(\boldsymbol{\xi}, \mathbf{x}) &= \sum_{\sigma \in \mathcal{S}_{n+1}} \mathcal{C}(\bar{\xi}_{\sigma_1}, \ldots, \bar{\xi}_{\sigma_{n+1}}) \exp(\mathrm{i}\bar{\xi}_{\sigma_1} x_1 + \cdots + \mathrm{i}\bar{\xi}_{\sigma_{n+1}} x_{n+1}), \\ \mathcal{C}(\bar{\xi}_1, \ldots, \bar{\xi}_{n+1}) &= \prod_{1 \leqslant j < k \leqslant n+1} \frac{\bar{\xi}_j - \bar{\xi}_k - \mathrm{i}g}{\bar{\xi}_j - \bar{\xi}_k} \end{aligned} \tag{4.1a}$$

(with $\mathcal{S}_{n+1}$ denoting the permutation group of $n + 1$ letters), and

$$\mathrm{e}^{\mathrm{i}\bar{\xi}_j} = \epsilon \prod_{\substack{1 \leqslant k \leqslant n+1 \\ k \neq j}} \frac{\mathrm{i}g + \bar{\xi}_j - \bar{\xi}_k}{\mathrm{i}g - \bar{\xi}_j + \bar{\xi}_k}, \qquad j = 1, \ldots, n+1, \tag{4.1b}$$

with $\epsilon^{n+1} = 1$ and

$$\bar{\xi}_j := \xi_j - \frac{1}{n+1}(\xi_1 + \cdots + \xi_{n+1}), \qquad j = 1, \ldots, n+1. \tag{4.1c}$$

The solutions of the Bethe equations are given by the global minima $\boldsymbol{\xi}_{\boldsymbol{\lambda}}, \boldsymbol{\lambda} \in \Lambda$, of the strictly convex functions

$$\begin{aligned} V_{\boldsymbol{\lambda}}(\boldsymbol{\xi}) = \frac{1}{2} \sum_{1 \leqslant j \leqslant n+1} \bar{\xi}_j^2 - 2\pi \sum_{1 \leqslant j \leqslant n+1} (\rho_j + \lambda_j) \bar{\xi}_j \\ + 2 \sum_{1 \leqslant j < k \leqslant n+1} \int_0^{\bar{\xi}_j - \bar{\xi}_k} \arctan\left(\frac{x}{g}\right) \mathrm{d}x, \end{aligned} \tag{4.2}$$

with $\rho_j = \frac{n}{2} + 1 - j$, $j = 1, \ldots, n+1$, and with $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_{n+1})$ running through the semi-lattice $\Lambda = \{k_1 \boldsymbol{\omega}_1 + \cdots + k_n \boldsymbol{\omega}_n \mid k_1, \ldots, k_n \in \mathbb{N}\}$, where $\boldsymbol{\omega}_j = (\mathbf{e}_1 + \cdots + \mathbf{e}_j) - \frac{j}{n+1}(\mathbf{e}_1 + \cdots + \mathbf{e}_{n+1})$, $j = 1, \ldots, n$.

The normalization conjecture amounts, in this situation, to the center-of-mass reduction of the Gaudin–Korepin integration formula [G2, K, E]

$$\frac{1}{\sqrt{n+1}} \int_{\mathbf{A}} |\Psi(\boldsymbol{\xi}_{\boldsymbol{\lambda}}, \mathbf{x})|^2 \,\mathrm{d}\mathbf{x} = [\mathcal{C}(\bar{\xi}_1, \ldots, \bar{\xi}_{n+1}) \mathcal{C}(-\bar{\xi}_1, \ldots, -\bar{\xi}_{n+1}) \det \mathcal{H}(\bar{\xi}_1, \ldots, \bar{\xi}_{n+1})]_{\boldsymbol{\xi} = \boldsymbol{\xi}_{\boldsymbol{\lambda}}}, \tag{4.3a}$$

with

$$\mathbf{A} = \{\mathbf{x} \in \mathbb{R}^{n+1} \mid x_1 > x_2 > \cdots > x_{n+1}, x_1 - x_{n+1} < 1, x_1 + \cdots + x_{n+1} = 0\}, \tag{4.3b}$$

and with $\mathcal{H}(\bar{\xi}_1, \ldots, \bar{\xi}_{n+1}) = [\mathcal{H}_{j,k}]_{1 \leqslant j,k \leqslant n+1}$ denoting the Hesse matrix with components

$$\mathcal{H}_{j,k} = \left(1 + \sum_{\ell=1}^{n+1} \frac{2g}{g^2 + (\bar{\xi}_j - \bar{\xi}_\ell)^2}\right)\delta_{j,k} - \frac{2g}{(g^2 + \bar{\xi}_j - \bar{\xi}_k)^2} \tag{4.3c}$$

(where $\delta_{j,k}$ refers to the Kronecker delta symbol).

For $n = 1$, this case amounts to the rank 1 situation at the end of the previous section. For $n = 2, 3$, we have performed an alternative check of this integration formula by confirming that

$$\sum_{\sigma,\sigma' \in \mathcal{S}_{n+1}} \frac{\mathcal{C}(\bar{\xi}_{\sigma_1}, \ldots, \bar{\xi}_{\sigma_{n+1}})}{\mathcal{C}(\bar{\xi}_{\sigma_1'}, \ldots, \bar{\xi}_{\sigma_{n+1}'})} \mathcal{R}(\bar{\xi}_{\sigma_1} - \bar{\xi}_{\sigma_1'}, \ldots, \bar{\xi}_{\sigma_{n+1}} - \bar{\xi}_{\sigma_{n+1}'}),$$

where

$$\mathcal{R}(\xi_1, \ldots, \xi_{n+1}) = \frac{1}{2\pi \mathrm{i}(n+1)} \oint_C \frac{(\mathrm{e}^z - 1)}{(z - \mathrm{i}\langle \boldsymbol{\omega}_1, \boldsymbol{\xi}\rangle) \cdots (z - \mathrm{i}\langle \boldsymbol{\omega}_n, \boldsymbol{\xi}\rangle)} \frac{\mathrm{d}z}{z},$$

simplifies to $\det \mathcal{H}(\bar{\xi}_1, \ldots, \bar{\xi}_{n+1})$ upon elimination of the exponentials $\mathrm{e}^{\mathrm{i}\bar{\xi}_1}, \ldots, \mathrm{e}^{\mathrm{i}\bar{\xi}_{n+1}}$ by means of the Bethe equations (4.1*b*).

4.2. *The case B_n ($n \geqslant 2$)*

The Bethe Ansatz wavefunction (2.5*a*) and the Bethe equations (2.5*c*) become

$$\begin{aligned}\Psi(\boldsymbol{\xi}, \mathbf{x}) &= \sum_{\substack{\sigma \in \mathcal{S}_n \\ \varepsilon_1,\ldots,\varepsilon_n \in \{1,-1\}}} \mathcal{C}(\varepsilon_1\xi_{\sigma_1}, \ldots, \varepsilon_n\xi_{\sigma_n}) \exp\left(\mathrm{i}\varepsilon_1\xi_{\sigma_1}x_1 + \cdots + \mathrm{i}\varepsilon_n\xi_{\sigma_n}x_n\right), \\ \mathcal{C}(\xi_1, \ldots, \xi_n) &= \prod_{1\leqslant j\leqslant n} \left(\frac{2\xi_j - \mathrm{i}g_1}{2\xi_j}\right) \prod_{1\leqslant j<k\leqslant n} \left(\frac{\xi_j - \xi_k - \mathrm{i}g_2}{\xi_j - \xi_k}\right)\left(\frac{\xi_j + \xi_k - \mathrm{i}g_2}{\xi_j + \xi_k}\right)\end{aligned} \tag{4.4a}$$

and

$$\mathrm{e}^{\mathrm{i}\xi_j} = \epsilon\left(\frac{\mathrm{i}g_1 + 2\xi_j}{\mathrm{i}g_1 - 2\xi_j}\right) \prod_{\substack{1\leqslant k\leqslant n \\ k\neq j}} \left(\frac{\mathrm{i}g_2 + \xi_j - \xi_k}{\mathrm{i}g_2 - \xi_j + \xi_k}\right)\left(\frac{\mathrm{i}g_2 + \xi_j + \xi_k}{\mathrm{i}g_2 - \xi_j - \xi_k}\right), \qquad j = 1, \ldots, n, \tag{4.4b}$$

with $\epsilon^2 = 1$.

The solutions of the Bethe equations are given by the global minima $\boldsymbol{\xi}_{\boldsymbol{\lambda}}$, $\boldsymbol{\lambda} \in \Lambda$, of

$$\begin{aligned}V_{\boldsymbol{\lambda}}(\boldsymbol{\xi}) = \frac{1}{2} \sum_{1\leqslant j\leqslant n} \xi_j^2 - 2\pi \sum_{1\leqslant j\leqslant n} (\rho_j + \lambda_j)\xi_j + \sum_{1\leqslant j\leqslant n} \int_0^{2\xi_j} \arctan\left(\frac{x}{g_1}\right) \mathrm{d}x \\ + 2 \sum_{1\leqslant j<k\leqslant n} \left(\int_0^{\xi_j-\xi_k} \arctan\left(\frac{x}{g_2}\right) \mathrm{d}x + \int_0^{\xi_j+\xi_k} \arctan\left(\frac{x}{g_2}\right) \mathrm{d}x\right),\end{aligned} \tag{4.5}$$

with $\rho_j = n + \frac{1}{2} - j$, $j = 1, \ldots, n$, and with $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ running through the semi-lattice $\Lambda = \{k_1\boldsymbol{\omega}_1 + \cdots + k_n\boldsymbol{\omega}_n \mid k_1, \ldots, k_n \in \mathbb{N}\}$, where $\boldsymbol{\omega}_j = \mathbf{e}_1 + \cdots + \mathbf{e}_j$, $j = 1, \ldots, n-1$, and $\omega_n = (\mathbf{e}_1 + \cdots + \mathbf{e}_n)/2$.

The normalization conjecture becomes

$$\frac{1}{2} \int_{\mathbf{A}} |\Psi(\boldsymbol{\xi}_{\boldsymbol{\lambda}}, \mathbf{x})|^2 \, \mathrm{d}\mathbf{x} = [\mathcal{C}(\xi_1, \ldots, \xi_n)\mathcal{C}(-\xi_1, \ldots, -\xi_n) \det \mathcal{H}(\xi_1, \ldots, \xi_n)]_{\boldsymbol{\xi}=\boldsymbol{\xi}_{\boldsymbol{\lambda}}}, \tag{4.6a}$$

with

$$\mathbf{A} = \{\mathbf{x} \in \mathbb{R}^n \mid x_1 > x_2 > \cdots > x_n > 0, x_1 + x_2 < 1\}, \tag{4.6b}$$

and with $\mathcal{H}(\xi_1,\ldots,\xi_n)=[\mathcal{H}_{j,k}]_{1\leqslant j,k\leqslant n}$ given by

$$\mathcal{H}_{j,k}=\left(1+\frac{4g_1}{g_1^2+4\xi_j^2}+\sum_{\ell=1}^{n}\left(\frac{2g_2}{g_2^2+(\xi_j-\xi_\ell)^2}+\frac{2g_2}{g_2^2+(\xi_j+\xi_\ell)^2}\right)\right)\delta_{j,k}$$
$$-\frac{2g_2}{g_2^2+(\xi_j-\xi_k)^2}-\frac{2g_2}{g_2^2+(\xi_j+\xi_k)^2}. \tag{4.6c}$$

For $n=2,3$, we have checked this integration formula by confirming that

$$\sum_{\substack{\sigma,\sigma'\in\mathcal{S}_n\\ \varepsilon_1,\ldots,\varepsilon_n,\varepsilon'_1,\ldots,\varepsilon'_n\in\{-1,1\}}}\frac{\mathcal{C}(\varepsilon_1\xi_{\sigma_1},\ldots,\varepsilon_n\xi_{\sigma_n})}{\mathcal{C}(\varepsilon'_1\xi_{\sigma'_1},\ldots,\varepsilon'_1\xi_{\sigma'_n})}\mathcal{R}(\varepsilon_1\xi_{\sigma_1}-\varepsilon'_1\xi_{\sigma'_1},\ldots,\varepsilon_n\xi_{\sigma_n}-\varepsilon'_n\xi_{\sigma'_n}),$$

where

$$\mathcal{R}(\xi_1,\ldots,\xi_n)=\frac{1}{4\pi\mathrm{i}}\oint_C\frac{(\mathrm{e}^z-1)}{(z-\mathrm{i}\langle\hat{\boldsymbol{\omega}}_1,\boldsymbol{\xi}\rangle)(2z-\mathrm{i}\langle\hat{\boldsymbol{\omega}}_2,\boldsymbol{\xi}\rangle)\cdots(2z-\mathrm{i}\langle\hat{\boldsymbol{\omega}}_n,\boldsymbol{\xi}\rangle)}\frac{\mathrm{d}z}{z}$$

with $\hat{\boldsymbol{\omega}}_j=\mathbf{e}_1+\cdots+\mathbf{e}_j$ $(j=1,\ldots,n)$, simplifies to $\det\mathcal{H}(\xi_1,\ldots,\xi_n)$ upon elimination of the exponentials $\mathrm{e}^{\mathrm{i}\xi_1},\ldots,\mathrm{e}^{\mathrm{i}\xi_n}$ by means of the Bethe equations (4.4*b*).

4.3. The case C_n $(n\geqslant 2)$

The Bethe Ansatz wavefunction (2.5*a*) and the Bethe equations (2.5*c*) become

$$\Psi(\boldsymbol{\xi},\mathbf{x})=\sum_{\substack{\sigma\in\mathcal{S}_n\\ \varepsilon_1,\ldots,\varepsilon_n\in\{1,-1\}}}\mathcal{C}(\varepsilon_1\xi_{\sigma_1},\ldots,\varepsilon_n\xi_{\sigma_n})\exp(\mathrm{i}\varepsilon_1\xi_{\sigma_1}x_1+\cdots+\mathrm{i}\varepsilon_n\xi_{\sigma_n}x_n),$$
$$\mathcal{C}(\xi_1,\ldots,\xi_n)=\prod_{1\leqslant j\leqslant n}\left(\frac{\xi_j-\mathrm{i}g_2}{\xi_j}\right)\prod_{1\leqslant j<k\leqslant n}\left(\frac{\xi_j-\xi_k-\mathrm{i}g_1}{\xi_j-\xi_k}\right)\left(\frac{\xi_j+\xi_k-\mathrm{i}g_1}{\xi_j+\xi_k}\right) \tag{4.7a}$$

and

$$\mathrm{e}^{i\xi_j}=\left(\frac{\mathrm{i}g_2+\xi_j}{\mathrm{i}g_2-\xi_j}\right)^2\prod_{\substack{1\leqslant k\leqslant n\\ k\neq j}}\left(\frac{\mathrm{i}g_1+\xi_j-\xi_k}{\mathrm{i}g_1-\xi_j+\xi_k}\right)\left(\frac{\mathrm{i}g_1+\xi_j+\xi_k}{\mathrm{i}g_1-\xi_j-\xi_k}\right),\qquad j=1,\ldots,n. \tag{4.7b}$$

The solutions of the Bethe equations are given by the global minima $\boldsymbol{\xi}_{\boldsymbol{\lambda}}$, $\boldsymbol{\lambda}\in\Lambda$, of

$$V_{\boldsymbol{\lambda}}(\boldsymbol{\xi})=\frac{1}{2}\sum_{1\leqslant j\leqslant n}\xi_j^2-2\pi\sum_{1\leqslant j\leqslant n}(\rho_j+\lambda_j)\xi_j+4\sum_{1\leqslant j\leqslant n}\int_0^{\xi_j}\arctan\left(\frac{x}{g_2}\right)\mathrm{d}x$$
$$+2\sum_{1\leqslant j<k\leqslant n}\left(\int_0^{\xi_j-\xi_k}\arctan\left(\frac{x}{g_1}\right)\mathrm{d}x+\int_0^{\xi_j+\xi_k}\arctan\left(\frac{x}{g_1}\right)\mathrm{d}x\right), \tag{4.8}$$

with $\rho_j=n+1-j$, $j=1,\ldots,n$, and with $\boldsymbol{\lambda}=(\lambda_1,\ldots,\lambda_n)$ running through the semi-lattice $\Lambda=\{k_1\boldsymbol{\omega}_1+\cdots+k_n\boldsymbol{\omega}_n\mid k_1,\ldots,k_n\in\mathbb{N}\}$, where $\boldsymbol{\omega}_j=\mathbf{e}_1+\cdots+\mathbf{e}_j$, $j=1,\ldots,n$.

The normalization conjecture becomes

$$\int_{\mathbf{A}}|\Psi(\boldsymbol{\xi}_{\boldsymbol{\lambda}},\mathbf{x})|^2\,\mathrm{d}\mathbf{x}=[\mathcal{C}(\xi_1,\ldots,\xi_n)\mathcal{C}(-\xi_1,\ldots,-\xi_n)\det\mathcal{H}(\xi_1,\ldots,\xi_n)]_{\boldsymbol{\xi}=\boldsymbol{\xi}_{\boldsymbol{\lambda}}}, \tag{4.9a}$$

with

$$\mathbf{A}=\left\{\mathbf{x}\in\mathbb{R}^n\;\middle|\;\tfrac{1}{2}>x_1>x_2>\cdots>x_n>0\right\}, \tag{4.9b}$$

and with $\mathcal{H}(\xi_1,\ldots,\xi_n) = [\mathcal{H}_{j,k}]_{1\leqslant j,k\leqslant n}$ given by

$$\mathcal{H}_{j,k} = \left(1 + \frac{4g_2}{g_2^2+\xi_j^2} + \sum_{\ell=1}^{n}\left(\frac{2g_1}{g_1^2+(\xi_j-\xi_\ell)^2} + \frac{2g_1}{g_1^2+(\xi_j+\xi_\ell)^2}\right)\right)\delta_{j,k} - \frac{2g_1}{g_1^2+(\xi_j-\xi_k)^2} - \frac{2g_1}{g_1^2+(\xi_j+\xi_k)^2}. \tag{4.9c}$$

For $n = 2, 3$, we have checked this integration formula by confirming that

$$\sum_{\substack{\sigma,\sigma'\in\mathcal{S}_n\\ \varepsilon_1,\ldots,\varepsilon_n,\varepsilon_1',\ldots,\varepsilon_n'\in\{-1,1\}}} \frac{\mathcal{C}\big(\varepsilon_1\xi_{\sigma_1},\ldots,\varepsilon_n\xi_{\sigma_n}\big)}{\mathcal{C}\big(\varepsilon_1'\xi_{\sigma_1'},\ldots,\varepsilon_1'\xi_{\sigma_n'}\big)}\mathcal{R}\big(\varepsilon_1\xi_{\sigma_1}-\varepsilon_1'\xi_{\sigma_1'},\ldots,\varepsilon_n\xi_{\sigma_n}-\varepsilon_n'\xi_{\sigma_n'}\big),$$

where

$$\mathcal{R}(\xi_1,\ldots,\xi_n) = \frac{1}{4\pi\mathrm{i}}\oint_C \frac{(\mathrm{e}^z-1)}{(2z-\mathrm{i}\langle\hat{\boldsymbol{\omega}}_1,\boldsymbol{\xi}\rangle)\cdots(2z-\mathrm{i}\langle\hat{\boldsymbol{\omega}}_{n-1},\boldsymbol{\xi}\rangle)(z-\mathrm{i}\langle\hat{\boldsymbol{\omega}}_n,\boldsymbol{\xi}\rangle)}\frac{\mathrm{d}z}{z}$$

with $\hat{\boldsymbol{\omega}}_j = \mathbf{e}_1+\cdots+\mathbf{e}_j$ $(j = 1,\ldots,n-1)$ and $\hat{\boldsymbol{\omega}}_n := (\mathbf{e}_1+\cdots+\mathbf{e}_n)/2$, simplifies to $\det\mathcal{H}(\xi_1,\ldots,\xi_n)$ upon elimination of the exponentials $\mathrm{e}^{i\xi_1},\ldots,\mathrm{e}^{i\xi_n}$ by means of the Bethe equations (4.7*b*).

4.4. *The case D_n $(n\geqslant 3)$*

The Bethe Ansatz wavefunction (2.5*a*) and the Bethe equations (2.5*c*) become (cf [G1])

$$\Psi(\boldsymbol{\xi},\mathbf{x}) = \sum_{\substack{\sigma\in\mathcal{S}_n\\ \varepsilon_1,\ldots,\varepsilon_n\in\{1,-1\}\\ \varepsilon_1\cdots\varepsilon_n=1}} \mathcal{C}\big(\varepsilon_1\xi_{\sigma_1},\ldots,\varepsilon_n\xi_{\sigma_n}\big)\exp\big(\mathrm{i}\varepsilon_1\xi_{\sigma_1}x_1+\cdots+\mathrm{i}\varepsilon_n\xi_{\sigma_n}x_n\big), \tag{4.10a}$$

$$\mathcal{C}(\xi_1,\ldots,\xi_n) = \prod_{1\leqslant j<k\leqslant n}\left(\frac{\xi_j-\xi_k-\mathrm{i}g}{\xi_j-\xi_k}\right)\left(\frac{\xi_j+\xi_k-\mathrm{i}g}{\xi_j+\xi_k}\right)$$

and

$$\mathrm{e}^{\mathrm{i}\xi_j} = \epsilon\prod_{\substack{1\leqslant k\leqslant n\\ k\neq j}}\left(\frac{\mathrm{i}g+\xi_j-\xi_k}{\mathrm{i}g-\xi_j+\xi_k}\right)\left(\frac{\mathrm{i}g+\xi_j+\xi_k}{\mathrm{i}g-\xi_j-\xi_k}\right), \qquad j = 1,\ldots,n, \tag{4.10b}$$

with $\epsilon^2 = 1$.

The solutions of the Bethe equations are given by the global minima $\boldsymbol{\xi}_{\boldsymbol{\lambda}}$, $\boldsymbol{\lambda}\in\Lambda$, of

$$V_{\boldsymbol{\lambda}}(\boldsymbol{\xi}) = \frac{1}{2}\sum_{1\leqslant j\leqslant n}\xi_j^2 - 2\pi\sum_{1\leqslant j\leqslant n}(\rho_j+\lambda_j)\xi_j + 2\sum_{1\leqslant j<k\leqslant n}\left(\int_0^{\xi_j-\xi_k}\arctan\left(\frac{x}{g}\right)\mathrm{d}x + \int_0^{\xi_j+\xi_k}\arctan\left(\frac{x}{g}\right)\mathrm{d}x\right), \tag{4.11}$$

with $\rho_j = n-j$, $j = 1,\ldots,n$, and with $\boldsymbol{\lambda} = (\lambda_1,\ldots,\lambda_n)$ running through the semi-lattice $\Lambda = \{k_1\boldsymbol{\omega}_1+\cdots+k_n\boldsymbol{\omega}_n \mid k_1,\ldots,k_n\in\mathbb{N}\}$, where $\boldsymbol{\omega}_j = \mathbf{e}_1+\cdots+\mathbf{e}_j$, $j = 1,\ldots,n-2$, and $\omega_{n-1} = (\mathbf{e}_1+\cdots+\mathbf{e}_{n-1}-\mathbf{e}_n)/2$, $\omega_n = (\mathbf{e}_1+\cdots+\mathbf{e}_n)/2$.

The normalization conjecture becomes

$$\frac{1}{2}\int_{\mathbf{A}}|\Psi(\boldsymbol{\xi}_{\boldsymbol{\lambda}},\mathbf{x})|^2\,\mathrm{d}\mathbf{x} = [\mathcal{C}(\xi_1,\ldots,\xi_n)\mathcal{C}(-\xi_1,\ldots,-\xi_n)\det\mathcal{H}(\xi_1,\ldots,\xi_n)]_{\boldsymbol{\xi}=\boldsymbol{\xi}_{\boldsymbol{\lambda}}}, \tag{4.12a}$$

with

$$\mathbf{A} = \{\mathbf{x} \in \mathbb{R}^n \mid x_1 > x_2 > \cdots > x_{n-1} > |x_n|, x_1 + x_2 < 1\}, \tag{4.12b}$$

and with $\mathcal{H}(\xi_1, \ldots, \xi_n) = [\mathcal{H}_{j,k}]_{1\leqslant j,k\leqslant n}$ given by

$$\begin{aligned}\mathcal{H}_{j,k} = &\left(1 + \sum_{\ell=1}^{n}\left(\frac{2g}{g^2 + (\xi_j - \xi_\ell)^2} + \frac{2g}{g^2 + (\xi_j + \xi_\ell)^2}\right)\right)\delta_{j,k} \\ &- \frac{2g}{g^2 + (\xi_j - \xi_k)^2} - \frac{2g}{g^2 + (\xi_j + \xi_k)^2}.\end{aligned} \tag{4.12c}$$

For $n = 3$, we have checked this integration formula by confirming that

$$\sum_{\substack{\sigma,\sigma'\in\mathcal{S}_n \\ \varepsilon_1,\ldots,\varepsilon_n,\varepsilon'_1,\ldots,\varepsilon'_n\in\{-1,1\} \\ \varepsilon_1\cdots\varepsilon_n,\varepsilon'_1\cdots\varepsilon'_n=1}} \frac{\mathcal{C}\big(\varepsilon_1\xi_{\sigma_1}, \ldots, \varepsilon_n\xi_{\sigma_n}\big)}{\mathcal{C}\big(\varepsilon'_1\xi_{\sigma'_1}, \ldots, \varepsilon'_1\xi_{\sigma'_n}\big)}\mathcal{R}\big(\varepsilon_1\xi_{\sigma_1} - \varepsilon'_1\xi_{\sigma'_1}, \ldots, \varepsilon_n\xi_{\sigma_n} - \varepsilon'_n\xi_{\sigma'_n}\big),$$

where

$$\mathcal{R}(\xi_1, \ldots, \xi_n) = \frac{1}{8\pi \mathrm{i}}\oint_C \frac{(\mathrm{e}^z - 1)}{(z - \mathrm{i}\langle\boldsymbol{\omega}_1, \boldsymbol{\xi}\rangle)(2z - \mathrm{i}\langle\boldsymbol{\omega}_2, \boldsymbol{\xi}\rangle)\cdots(2z - \mathrm{i}\langle\boldsymbol{\omega}_{n-2}, \boldsymbol{\xi}\rangle)(z - \mathrm{i}\langle\boldsymbol{\omega}_{n-1}, \boldsymbol{\xi}\rangle)(z - \mathrm{i}\langle\boldsymbol{\omega}_n, \boldsymbol{\xi}\rangle)}\frac{\mathrm{d}z}{z},$$

simplifies to $\det \mathcal{H}(\xi_1, \ldots, \xi_n)$ upon elimination of the exponentials $\mathrm{e}^{\mathrm{i}\xi_1}, \ldots, \mathrm{e}^{\mathrm{i}\xi_n}$ by means of the Bethe equations (4.10*b*).

4.5. The case G_2

The Bethe Ansatz wavefunction (2.5*a*) and the Bethe equations (2.5*c*) become (cf [CY])

$$\Psi(\boldsymbol{\xi}, \mathbf{x}) = \sum_{\substack{\sigma\in\mathcal{S}_3 \\ \varepsilon\in\{1,-1\}}} \mathcal{C}\big(\varepsilon\bar{\xi}_{\sigma_1}, \varepsilon\bar{\xi}_{\sigma_2}, \varepsilon\bar{\xi}_{\sigma_3}\big)\exp\big(\mathrm{i}\varepsilon\big(\bar{\xi}_{\sigma_1}x_1 + \bar{\xi}_{\sigma_2}x_2 + \bar{\xi}_{\sigma_3}x_3\big)\big),$$

$$\begin{aligned}\mathcal{C}(\bar{\xi}_1, \bar{\xi}_2, \bar{\xi}_3) = &\left(\frac{\bar{\xi}_1 - \bar{\xi}_2 - \mathrm{i}g_1}{\bar{\xi}_1 - \bar{\xi}_2}\right)\left(\frac{-\bar{\xi}_1 + \bar{\xi}_3 - \mathrm{i}g_1}{-\bar{\xi}_1 + \bar{\xi}_3}\right)\left(\frac{-\bar{\xi}_2 + \bar{\xi}_3 - \mathrm{i}g_1}{-\bar{\xi}_2 + \bar{\xi}_3}\right) \\ &\times\left(\frac{-\bar{\xi}_1 - \mathrm{i}g_2}{-\bar{\xi}_1}\right)\left(\frac{-\bar{\xi}_2 - \mathrm{i}g_2}{-\bar{\xi}_2}\right)\left(\frac{\bar{\xi}_3 - \mathrm{i}g_2}{\bar{\xi}_3}\right)\end{aligned} \tag{4.13a}$$

and

$$\begin{aligned}\mathrm{e}^{\mathrm{i}\bar{\xi}_1} &= \left(\frac{\mathrm{i}g_2 + \bar{\xi}_1}{\mathrm{i}g_2 - \bar{\xi}_1}\right)^2\left(\frac{\mathrm{i}g_2 - \bar{\xi}_2}{\mathrm{i}g_2 + \bar{\xi}_2}\right)\left(\frac{\mathrm{i}g_2 - \bar{\xi}_3}{\mathrm{i}g_2 + \bar{\xi}_3}\right)\left(\frac{\mathrm{i}g_1 + \bar{\xi}_1 - \bar{\xi}_2}{\mathrm{i}g_1 - \bar{\xi}_1 + \bar{\xi}_2}\right)\left(\frac{\mathrm{i}g_1 + \bar{\xi}_1 - \bar{\xi}_3}{\mathrm{i}g_1 - \bar{\xi}_1 + \bar{\xi}_3}\right), \\ \mathrm{e}^{\mathrm{i}\bar{\xi}_2} &= \left(\frac{\mathrm{i}g_2 + \bar{\xi}_2}{\mathrm{i}g_2 - \bar{\xi}_2}\right)^2\left(\frac{\mathrm{i}g_2 - \bar{\xi}_1}{\mathrm{i}g_2 + \bar{\xi}_1}\right)\left(\frac{\mathrm{i}g_2 - \bar{\xi}_3}{\mathrm{i}g_2 + \bar{\xi}_3}\right)\left(\frac{\mathrm{i}g_1 + \bar{\xi}_2 - \bar{\xi}_1}{\mathrm{i}g_1 - \bar{\xi}_2 + \bar{\xi}_1}\right)\left(\frac{\mathrm{i}g_1 + \bar{\xi}_2 - \bar{\xi}_3}{\mathrm{i}g_1 - \bar{\xi}_2 + \bar{\xi}_3}\right), \\ \mathrm{e}^{i\bar{\xi}_3} &= \left(\frac{\mathrm{i}g_2 + \bar{\xi}_3}{\mathrm{i}g_2 - \bar{\xi}_3}\right)^2\left(\frac{\mathrm{i}g_2 - \bar{\xi}_1}{\mathrm{i}g_2 + \bar{\xi}_1}\right)\left(\frac{\mathrm{i}g_2 - \bar{\xi}_2}{\mathrm{i}g_2 + \bar{\xi}_2}\right)\left(\frac{\mathrm{i}g_1 + \bar{\xi}_3 - \bar{\xi}_1}{\mathrm{i}g_1 - \bar{\xi}_3 + \bar{\xi}_1}\right)\left(\frac{\mathrm{i}g_1 + \bar{\xi}_3 - \bar{\xi}_2}{\mathrm{i}g_1 - \bar{\xi}_3 + \bar{\xi}_2}\right),\end{aligned} \tag{4.13b}$$

where

$$\bar{\xi}_1 := \tfrac{1}{3}(2\xi_1 - \xi_2 - \xi_3), \quad \bar{\xi}_2 := \tfrac{1}{3}(-\xi_1 + 2\xi_2 - \xi_3), \quad \bar{\xi}_3 := \tfrac{1}{3}(-\xi_1 - \xi_2 + 2\xi_3). \tag{4.13c}$$

The solutions of the Bethe equations are given by the global minima $\boldsymbol{\xi}_{\boldsymbol{\lambda}}, \boldsymbol{\lambda} \in \Lambda$, of

$$\begin{aligned}V_{\boldsymbol{\lambda}}(\boldsymbol{\xi}) = &\frac{1}{2}\sum_{1\leqslant j\leqslant 3}\bar{\xi}_j^2 - 2\pi\sum_{1\leqslant j\leqslant 3}(\rho_j + \lambda_j)\bar{\xi}_j \\ &+ 2\sum_{1\leqslant j<k\leqslant 3}\int_0^{\bar{\xi}_j - \bar{\xi}_k}\arctan\left(\frac{x}{g_1}\right)\mathrm{d}x + 6\sum_{1\leqslant j\leqslant 3}\int_0^{\bar{\xi}_j}\arctan\left(\frac{x}{g_2}\right)\mathrm{d}x,\end{aligned} \tag{4.14}$$

with $(\rho_1, \rho_2, \rho_3) = (-1, -2, 3)$ and with $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ running through the semi-lattice $\Lambda = \{k_1\boldsymbol{\omega}_1 + k_2\boldsymbol{\omega}_2 \mid k_1, k_2 \in \mathbb{N}\}$, where $\boldsymbol{\omega}_1 = -\mathbf{e}_2 + \mathbf{e}_3$, $\boldsymbol{\omega}_2 = -\mathbf{e}_1 - \mathbf{e}_2 + 2\mathbf{e}_3$.

The normalization conjecture becomes

$$\int_{\mathbf{A}} |\Psi(\boldsymbol{\xi}_{\boldsymbol{\lambda}}, \mathbf{x})|^2 \,\mathrm{d}\mathbf{x} = [\mathcal{C}(\bar{\xi}_1, \bar{\xi}_2, \bar{\xi}_3)\mathcal{C}(-\bar{\xi}_1, -\bar{\xi}_2, -\bar{\xi}_3) \det \mathcal{H}(\bar{\xi}_1, \bar{\xi}_2, \bar{\xi}_3)]_{\boldsymbol{\xi}=\boldsymbol{\xi}_{\boldsymbol{\lambda}}}, \qquad (4.15a)$$

with

$$\mathbf{A} = \{\mathbf{x} \in \mathbb{R}^3 \mid x_1 > x_2, x_2 + x_3 > 2x_1, 1 + x_1 + x_2 > 2x_3, x_1 + x_2 + x_3 = 0\}, \qquad (4.15b)$$

and with $\mathcal{H}(\bar{\xi}_1, \bar{\xi}_2, \bar{\xi}_3) = [\mathcal{H}_{j,k}]_{1\leqslant j,k\leqslant 3}$ given by

$$\mathcal{H}_{1,1} = 1 + \frac{2g_1}{g_1^2 + (\bar{\xi}_1 - \bar{\xi}_2)^2} + \frac{2g_1}{g_1^2 + (\bar{\xi}_1 - \bar{\xi}_3)^2} + \frac{\frac{8}{3}g_2}{g_2^2 + \bar{\xi}_1^2} + \frac{\frac{2}{3}g_2}{g_2^2 + \bar{\xi}_2^2} + \frac{\frac{2}{3}g_2}{g_2^2 + \bar{\xi}_3^2}$$

$$\mathcal{H}_{2,2} = 1 + \frac{2g_1}{g_1^2 + (\bar{\xi}_2 - \bar{\xi}_1)^2} + \frac{2g_1}{g_1^2 + (\bar{\xi}_2 - \bar{\xi}_3)^2} + \frac{\frac{2}{3}g_2}{g_2^2 + \bar{\xi}_1^2} + \frac{\frac{8}{3}g_2}{g_2^2 + \bar{\xi}_2^2} + \frac{\frac{2}{3}g_2}{g_2^2 + \bar{\xi}_3^2}$$

$$\mathcal{H}_{3,3} = 1 + \frac{2g_1}{g_1^2 + (\bar{\xi}_3 - \bar{\xi}_1)^2} + \frac{2g_1}{g_1^2 + (\bar{\xi}_3 - \bar{\xi}_2)^2} + \frac{\frac{2}{3}g_2}{g_2^2 + \bar{\xi}_1^2} + \frac{\frac{2}{3}g_2}{g_2^2 + \bar{\xi}_2^2} + \frac{\frac{8}{3}g_2}{g_2^2 + \bar{\xi}_3^2}$$

$$\mathcal{H}_{1,2} = \mathcal{H}_{2,1} = -\frac{2g_1}{g_1^2 + (\bar{\xi}_1 - \bar{\xi}_2)^2} - \frac{\frac{4}{3}g_2}{g_2^2 + \bar{\xi}_1^2} - \frac{\frac{4}{3}g_2}{g_2^2 + \bar{\xi}_2^2} + \frac{\frac{2}{3}g_2}{g_2^2 + \bar{\xi}_3^2}$$

$$\mathcal{H}_{1,3} = \mathcal{H}_{3,1} = -\frac{2g_1}{g_1^2 + (\bar{\xi}_1 - \bar{\xi}_3)^2} - \frac{\frac{4}{3}g_2}{g_2^2 + \bar{\xi}_1^2} + \frac{\frac{2}{3}g_2}{g_2^2 + \bar{\xi}_2^2} - \frac{\frac{4}{3}g_2}{g_2^2 + \bar{\xi}_3^2}$$

$$\mathcal{H}_{2,3} = \mathcal{H}_{3,2} = -\frac{2g_1}{g_1^2 + (\bar{\xi}_2 - \bar{\xi}_3)^2} + \frac{\frac{2}{3}g_2}{g_2^2 + \bar{\xi}_1^2} - \frac{\frac{4}{3}g_2}{g_2^2 + \bar{\xi}_2^2} - \frac{\frac{4}{3}g_2}{g_2^2 + \bar{\xi}_3^2}.$$

We have checked this integration formula by confirming that

$$\sum_{\substack{\sigma,\sigma' \in \mathcal{S}_3 \\ \varepsilon,\varepsilon' \in \{1,-1\}}} \frac{\mathcal{C}\big(\varepsilon\bar{\xi}_{\sigma_1}, \varepsilon\bar{\xi}_{\sigma_2}, \varepsilon\bar{\xi}_{\sigma_3}\big)}{\mathcal{C}\big(\varepsilon'\bar{\xi}_{\sigma'_1}, \varepsilon'\bar{\xi}_{\sigma'_2}, \varepsilon'\bar{\xi}_{\sigma'_3}\big)} \mathcal{R}\big(\varepsilon\bar{\xi}_{\sigma_1} - \varepsilon'\bar{\xi}_{\sigma'_1}, \varepsilon\bar{\xi}_{\sigma_2} - \varepsilon'\bar{\xi}_{\sigma'_2}, \varepsilon\bar{\xi}_{\sigma_3} - \varepsilon'\bar{\xi}_{\sigma'_3}\big),$$

where

$$\mathcal{R}(\xi_1, \xi_2, \xi_3) = \frac{1}{2\pi \mathrm{i}} \oint_C \frac{(\mathrm{e}^z - 1)}{(3z - \mathrm{i}\langle\hat{\boldsymbol{\omega}}_1, \boldsymbol{\xi}\rangle)(2z - \mathrm{i}\langle\hat{\boldsymbol{\omega}}_2, \boldsymbol{\xi}\rangle)} \frac{\mathrm{d}z}{z},$$

with $\hat{\boldsymbol{\omega}}_1 = -\mathbf{e}_2 + \mathbf{e}_3$ and $\hat{\boldsymbol{\omega}}_2 = (-\mathbf{e}_1 - \mathbf{e}_2 + 2\mathbf{e}_3)/3$, simplifies to $\det \mathcal{H}(\bar{\xi}_1, \bar{\xi}_2, \bar{\xi}_3)$ upon elimination of the exponentials $\mathrm{e}^{\mathrm{i}\bar{\xi}_1}, \mathrm{e}^{\mathrm{i}\bar{\xi}_2}, \mathrm{e}^{\mathrm{i}\bar{\xi}_3}$ by means of the Bethe equations (4.13*b*).

Remark. The above expressions correspond to the canonical choice of the positive roots in accordance with Bourbaki [B]. However, for the root system G_2 our formulae become a bit more elegant when picking instead the simple roots as $\boldsymbol{\alpha}_1 = \mathbf{e}_1 - \mathbf{e}_2$, $\boldsymbol{\alpha}_2 = -\mathbf{e}_1 + 2\mathbf{e}_2 - \mathbf{e}_3$. This amounts to interchanging the first two coordinates and flipping the sign of all three coordinates.

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